A Note on Preservers of Decomposability*

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ABSTRACT

Let V be a vector space over the algebraically closed field K. We prove the existence of a finite set of polynomials of m^n indeterminates $(F_i)_{i \in I}$ such that if $\{e_1, \dots, e_n\}$ is a basis of V and $\{e_\alpha^\otimes | \alpha \in \Gamma_{m,n}\}$ is the basis induced in $\otimes^m V$ by the basis $\{e_1, \dots, e_n\}$, then

$$z = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha e_\alpha^\otimes \in \otimes^m V$$

is decomposable symmetrized (with $\lambda = 1$) if and only if $F_i(a_\alpha \,|\, \alpha \in \Gamma_{m,n}) = 0$, $i \in I$. We use these results to prove that the concept of decomposability preserver coincides with the concept of m-decomposability preserver (a linear operator that maps star products of linearly independent vectors into decomposable symmetrized tensors).

INTRODUCTION

Let V be a vector space of dimension n over an algebraically closed field K of characteristic zero. In $\otimes^m V$ (the mth tensor power of V), two subspaces are often considered: the Grassmann space, $\wedge^m V$, and the space

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of completely symmetric tensors, $\vee^m V$. The Grassmann space has been especially studied in virtue of its connection with geometry. It is known that many results about this subspace have parallels in the space of completely symmetric tensors of degree m or in some other generalizations of the m-tensor power of a vector space (symmetry classes of tensors).

The aim of this article is to extend to some symmetry classes of tensors important results already stated in $\wedge^m V$, namely the characterization of the decomposable symmetrized (with $\lambda = 1$) tensors by means of the Plücker polynomials [17] and to get from this a property of the preservers of decomposability. We are going to prove the existence of a finite set of polynomials of m^n indeterminates $(F_i)_{i\in I}$ such that if $\{e_1, \dots, e_n\}$ is a basis of V and $\{e_\alpha^\otimes | \alpha \in \Gamma_{m,n}\}$ is the basis induced in $\otimes^m V$ by the basis $\{e_1, \dots, e_n\}$ then

$$z = \sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes \in \otimes^m V$$

is decomposable symmetrized if and only if $F_i(a_\alpha \mid \alpha \in \Gamma_{m,n}) = 0$, $i \in I$. We will use these results to prove that the concept of decomposability preserver coincides with the concept of m-decomposability preserver (a linear operator that maps star products of linearly independent vectors into decomposable symmetrized tensors).

PRELIMINARIES

Let $\sigma \in S_m$. We denote by $P(\sigma)$ the unique linear operator from $\otimes^m V$ into $\otimes^m V$ satisfying

$$P(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}$$

for all $x_1, \dots, x_m$ in V.

If $c : S_m \to K$ is an arbitrary function, the symmetrizer T_c is defined as the linear combination of the operators $P(\sigma)$ such that the coefficient of $P(\sigma)$ is $c(\sigma)$, i.e.,

$$T_c = \sum_{\sigma\in S_m} c(\sigma)P(\sigma).$$

The range of T_c is called the symmetry class of tensors and denoted by $V(c)$ (obviously $\otimes^m V$ is a symmetry class of tensors). The image by T_c of the decomposable tensor $x_1 \otimes \cdots \otimes x_m$ is denoted by $x_1 * \cdots * x_m$ and is called the star product of the tensors $x_1, \dots, x_m$ or the decomposable symmetrized tensor.

We denote by $\mathbb{P}^n(K)$ (or $\mathbb{P}^n$) the K-projective space of dimension n. If $x = \langle (x_0, \dots, x_n) \rangle \in \mathbb{P}^n(K)$, then $0 \neq (x_0, \dots, x_n)$ is called a system of coordinates of x.

We are going to work in this paper with projective varieties. With the aim of self-completeness we quote the results and concepts of algebraic geometry used in this article.

The product of the projective spaces $\mathbb{P}^m, \mathbb{P}^n$ is defined (cf. [20, pp. 42–43]) as a pair $(\varphi, \mathbb{P}^N)$ such that

$$\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(m+1)(n+1)-1}$$

and $\varphi(\langle x \rangle, \langle y \rangle) = \langle x \otimes y \rangle$. Often the range of φ is called the product of $\mathbb{P}^m$ and $\mathbb{P}^n$. It is not difficult to see that the product of two projective spaces is an irreducible projective variety.

DEFINITION [20]. Let X be an irreducible projective variety. A *regular mapping* $f : X \to \mathbb{P}^m$ is a function such that for all $x \in X$ there exists a family of homogeneous polynomials of the same degree $F_0, \dots, F_s$ satisfying:

$$f(\langle x \rangle) = \langle (F_0(x), \dots, F_m(x)) \rangle. \tag{i}$$

Observe that there must exist an integer $i \in \{1, \dots, m\}$ such that $F_i(x) \neq 0$.

THEOREM A [20, p. 45]. *If X is a projective variety and f a regular mapping from X into $\mathbb{P}^m$, then $f(X)$ is a closed projective set.*

We denote by $\Gamma_{m,n}$ the set of the maps from $\{1, \dots, m\}$ into $\{1, \dots, n\}$. The subset of the strictly increasing functions of $\Gamma_{m,n}$ is denoted by $Q_{m,n}$. It is well known that if V is a vector space and $E = \{e_1, \dots, e_n\}$ is a basis of V, then

$$\mathscr{E} = \left\{ e_\alpha^\otimes = e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} \middle| \alpha \in \Gamma_{m,n} \right\}$$

is a basis of $\otimes^m V$. If $(f_1, \dots, f_n)$ is the dual basis of $(e_1, \dots, e_n)$, then

$$\mathscr{F} = \left\{ f_\alpha^\otimes = f_{\alpha(1)} \otimes \cdots \otimes f_{\alpha(m)} \middle| \alpha \in \Gamma_{m,n} \right\}.$$

is the dual basis of $\mathscr{E}$. We are assuming both bases are lexicographically ordered.

If $z \in \otimes^m V$, we denote by $z_{\mathscr{E}}$ the n^m-tuple of the components of z in the basis $\mathscr{E}$ (ordered lexicographically).

If $A = (a_{ij})$ is an $m \times n$ matrix over the field K and $\alpha \in \Gamma_{m,n}$, we denote by

$$A[1, \dots, m|\alpha]$$

the $m \times m$ matrix whose jth column is column $\alpha(j)$ of A, $j = 1, \dots, m$; i.e., the (i, j) entry of A is $a_{i, \alpha(j)}$, $i, j = 1, \dots, m$. Similarly, if $r \leqslant \min\{m, n\}$ and $\alpha \in \Gamma_{r,m}$ and $\beta \in \Gamma_{r,n}$, we denote by $A[\alpha|\beta]$ the $r \times r$ matrix whose (i, j) entry is $a_{\alpha(i), \beta(j)}$.

If $B = (b_{ij})$ is an $m \times m$ matrix over K, and c is, as before, an arbitrary function from S_m into K, we denote by $d_c(B)$ the element of K given by

$$d_c(B) = \sum_{\sigma \in S_m} c(\sigma) \prod_{t=1}^{m} b_{t, \sigma(t)}.$$

Let G be subgroup of S_m, and λ an irreducible K-character of G. We will identify λ with the extension of λ to S_m that is zero in $S_m \setminus G$. In this case we denote the linear operator $T_{(\lambda(\mathrm{id})/|G|\lambda)}$ by $T(G, \lambda)$, and the symmetry class of tensors $T(G, \lambda)(\otimes^m V)$ by $V_\lambda(G)$. The order of the group G will be denoted by $|G|$.

THE PROBLEM

The question we are going to solve in this article is concerned with new types of preservers. In fact, in [1–4, 7, 10–15, 18, 19, 21–24] linear preservers of decomposability are studied. The purpose of these papers is to characterize the linear operators on $\otimes^m V$ that map decomposable tensors into decomposable tensors (decomposable symmetrized tensors into decomposable symmetrized tensors). For the "aficionado" we mention the excellent bibliography of R. Grone, S. Pierce, C. K. Li, and N. K. Tsing [9].

Definition. Let $\mathscr{L} \in L(\otimes^m V; \otimes^m V)$. We say that $\mathscr{L}$ preserves decomposability (preserves symmetrized decomposability) if the image by $\mathscr{L}$ of a decomposable tensor (decomposable symmetrized tensor) is decomposable (decomposable symmetrized).

We say that a linear operator $\mathscr{L}$ of $\otimes^m V$, is a *t-decomposability preserver* with respect to $V(c)$ if the image of any decomposable symmetrized tensor $T_c(x_1 \otimes \cdots \otimes x_m)$ such that $\dim\langle x_1, \dots, x_m\rangle = t$ (t-decomposable symmetrized tensor) is a decomposable symmetrized tensor. It is obvious that $\mathscr{L}$ is a linear decomposability preserver if and only if it is a t-decomposability preserver for every t belonging to $\{1, \dots, m\}$. It can be seen that $\mathscr{L}$ can be a t-decomposability preserver for some t without preserving decomposability. In fact, let $m = 2$, $t = 1$, and $\mathscr{L} = P(\mathrm{id}) + P(12)$. It is obvious that $\mathscr{L}$ is a 1-decomposability preserver. However, if x_1, x_2 are linearly independent vectors, then $\mathscr{L}(x_1 \otimes x_2) = x_1 \otimes x_2 + x_2 \otimes x_1$, which is a nondecomposable tensor.

Our main result is concerned with relations between the t-preservers for different values of t.

GRASSMANNIAN-TYPE VARIETIES

It is well known that if we denote the alternating character of S_m by ϵ, then the Grassmann space of degree m associated to V is the range of $T(S_m, \epsilon)$, i.e.,

$$T(S_m, \epsilon)(\otimes^m V) = \wedge^m V.$$

Moreover,

$$T(S_m, \epsilon)(v_1 \otimes \cdots \otimes v_m) = v_1 \wedge \cdots \wedge v_m.$$

Let $\Gamma_{m,n} = \{\alpha_0, \dots, \alpha_{n^m-1}\}$, with $\alpha_0 < \cdots < \alpha_{n^m-1}$ by the lexicographic order. Let X_{α_i} be indeterminates, $1 \leqslant i \leqslant n^m - 1$. It is well known [17] that there exists a family of homogeneous polynomials

$$G_\nu \in K\left[X_{\alpha_0}, \dots, X_{\alpha_{n^m-1}}\right], \qquad \nu = 1, \dots, s,$$

such that

$$0 \neq z \in \wedge^m V \text{ is a decomposable symmetrized tensor}$$

$$\Leftrightarrow \quad G_\nu(z_{\mathscr{E}}) = 0, \quad \nu = 1, \dots, s.$$

Since when $z \neq 0$, z is a decomposable symmetrized tensor if and only if az ($a \in K \setminus \{0\}$) is a decomposable symmetrized tensor, we can conclude that symmetrized decomposability is a property which we can express in terms of the projective space $\mathbb{P}^{n^m-1}$ by saying that the tensor z is decomposable symmetrized if and only if $\langle z_{\mathscr{G}} \rangle$ belongs to the projective variety defined by the homogeneous polynomials $(G_\nu)_{\nu=1,\dots,s}$. This variety is known as the *Grassmannian of indices m and n*. Since we can identify the subspaces of dimension m of V by nonzero decomposable symmetrized tensors $\wedge^m V$, we can say that the Grassmannian of indices m and n is the variety of the subspaces of dimension m of a vector space of dimension n over K.

Using the geometrical results presented above, it is possible to state a result on the existence of a Grassmannian-type variety for decomposable symmetrized tensors associated to symmetrizers of the form $T(G, 1)$.

LEMMA. *Let V be a vector space over the field K, and $E = \{e_1, \dots, e_n\}$ a basis of V. Assume that*

$$x_i = \sum_{j=1}^{n} a_{ij} e_j, \qquad i = 1, \dots, m,$$

are m elements of V. Then the following properties are true:

(1) *If* $\alpha \in \Gamma_{m,n}$, *the* $e_\alpha^\otimes$*-component of* $T(G, 1)(x_1 \otimes \cdots \otimes x_m) = x_1 * \cdots * x_m$ *is*

$$\frac{1}{|G|} d_1^G A[1, \dots, m | \alpha].$$

(2) *The decomposable symmetrized tensor $x_1 * \cdots * x_m$ is zero if and only if $x_1 \otimes \cdots \otimes x_m$ is zero.*

Proof. (1): The $e_\alpha^\otimes$-component of $x_1 * \cdots * x_m$ is

$$f_\alpha^\otimes(x_1 * \cdots * x_m) = \frac{1}{|G|} f_\alpha^\otimes\left(\sum_{\sigma \in G} x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)} \right)$$
$$= \frac{1}{|G|} d_1^G A[1, \dots, m | \alpha].$$

(2): The proof of (2) can be found in [16]. ∎

REMARK 1. Statement (1) of the Lemma says, in particular, that the mapping φ given by

$$\langle (x_1 \otimes \cdots \otimes x_m)_{\mathscr{G}} \rangle \to \langle (x_1 * \cdots * x_m)_{\mathscr{G}} \rangle$$

has, as component functions, homogeneous polynomial functions $(F_\alpha)_{\alpha \in \Gamma_{m,n}}$ of degree 1, where

$$F_\alpha(X_{\alpha_0}, \ldots, X_{\alpha_{n^m-1}}) = \frac{1}{|G|} \sum_{\sigma \in G} X_{\alpha\sigma}.$$

In fact,

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha \in \Gamma_{m,n}} \prod_{t=1}^{m} a_{t,\alpha(t)} e_\alpha^\otimes.$$

Then

$$x_1 * \cdots * x_m = \sum_{\alpha \in \Gamma_{m,n}} \left(\frac{1}{|G|} \sum_{\sigma \in G} \prod_{t=1}^{m} a_{t,\alpha\sigma(t)} \right) e_\alpha^\otimes.$$

Denoting by a_α the element $\prod_{t=1}^m a_{t,\alpha\sigma(t)}$ of K, we can derive from the former equality that

$$x_1 * \cdots * x_m = \sum_{\alpha \in \Gamma_{m,n}} F_\alpha(a_{\alpha_0}, \ldots, a_{\alpha_{n^m-1}}) e_\alpha^\otimes.$$

Since we have already seen that if $x_1 \otimes \cdots \otimes x_m \neq 0$ then $x_1 * \cdots * x_m \neq 0$, we can conclude that φ is regular and thus a continuous function, considering fixed the Zariski topology in both the domain and the arrival set of φ [20, p. 38] (bear in the mind that the product of m factors of the projective space of dimension $n - 1$ is an irreducible projective variety which is, by definition, the set of the subspaces of $\otimes^m K^n$ of dimension 1 generated by decomposable tensors). Since the image by a continuous function of an irreducible set is irreducible [5, p. 186], Theorem A allows us to conclude the theorem

THEOREM 1. *The set*

$$\{\langle z_{\mathscr{g}}\rangle \mid z = x_1 * \cdots * x_m = T(G,1)(x_1 \otimes \cdots \otimes x_m),\ x_1, \ldots, x_m \in V \setminus \{0\}\}$$

is an irreducible projective closed set in $\mathbb{P}^{n^m-1}$.

REMARK. R. Grone has published a result similar to Theorem 1 (in fact, in a certain sense a more general result) in [8]. However, his proof does not work, since it is based on the argument that a linear operator image of an algebraic variety is an algebraic variety, which is not true, as we can see by taking the hypersurface $xy = 1$ and the projection on the x-axis.

PRESERVERS

Our main result states that the notion of r-decomposability preserver and the notion of t-decomposability preserver for all t, $1 \leqslant t \leqslant r$, coincide, i.e.:

THEOREM 2. *Let G be a group, and $\mathscr{L}$ be a linear operator from $L(\otimes^m V; \otimes^m V)$. The linear operator $\mathscr{L}$ is an r-decomposability preserver with respect to $V_1(G)$ if and only if it is a t-decomposability preserver with respect to $V_1(G)$ for $1 \leqslant t \leqslant r$.*

Proof. Consider the mapping φ from $K^{m\times r} \times K^{r \times n}$ into $K^{m \times n}$ whose (i, j)th coordinate function is $\sum_{1 \leqslant j \leqslant n} X_{ik} Y_{kj}$. The mapping φ is a regular mapping and thus continuous (for the Zariski topology), and its range is the set of $m \times n$ matrices of rank less than or equal to r. Therefore the set W of the $m \times n$ matrices over K of rank less than or equal to r is an irreducible affine closed set. We are going to prove that if $\mathscr{L}$ preserves r-decomposability, it preserves t-decomposability for $1 \leqslant t \leqslant r$.

Assume that V has a basis $\{e_1, \dots, e_n\}$. Let $Y_{11}, Y_{12}, \dots, Y_{1n}, Y_{21}, \dots, Y_{2n}, \dots, Y_{m1}, \dots, Y_{mn}$ be mn distinct indeterminates, and $Y = (Y_{ij})$ the $m \times n$ matrix of the indeterminates. If $\alpha \in \Gamma_{m,n}$, we define $H_\alpha \in K[Y_{11}, \dots, Y_{1n}, Y_{21}, \dots, Y_{2n}, \dots, Y_{m1}, \dots, Y_{mn}]$ by

$$H_\alpha(Y_{11}, \dots, Y_{1n}, Y_{21}, \dots, Y_{2n}, \dots, Y_{m1}, \dots, Y_{mn}) = \frac{1}{|G|} d_1^G Y[1, \dots, m | \alpha].$$

We denote by $\hat{\mathscr{L}} = (\hat{\mathscr{L}}_{\alpha\beta})$ the matrix of the linear operator $\mathscr{L}$ in the basis induced in $\otimes^m V$ by the basis $\{e_1, \dots, e_n\}$ of V. Let H'_α be the polynomial of $K[Y_{11}, \dots, Y_{1n}, Y_{21}, \dots, Y_{2n}, \dots, Y_{m1}, \dots, Y_{mn}]$,

$$H'_\alpha(Y_{11}, \dots, Y_{1n}, \dots, Y_{m1}, \dots, Y_{mn})$$

$$= \sum_{\beta \in \Gamma'_{m,n}} \hat{\mathscr{L}}_{\alpha\beta} H_\beta(Y_{11}, \dots, Y_{1n}, \dots, Y_{m1}, \dots, Y_{mn}).$$

It is an easy consequence of the Lemma that the tensor

$$z = \sum_{\alpha \in \Gamma_{m,n}} c_\alpha e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)}$$

is a decomposable symmetrized tensor if and only if there is a solution

$$(\xi_{ij})_{i \in \{1,\ldots,m\},\, j \in \{1,\ldots,n\}},$$

of the system

$$H_\alpha(Y_{11},\ldots,Y_{1n},Y_{21},\ldots,Y_{2n},\ldots,Y_{m1},\ldots,Y_{mn}) = c_\alpha, \qquad \alpha \in \Gamma_{m,n}. \quad (1)$$

On the other hand z is a star product of vectors whose linear closure has dimension r if there is a solution of the system (1) which is not a solution of the system

$$\det Y[\alpha|\beta] = D_{\alpha,\beta}(Y_{11},\ldots,Y_{1n},Y_{21},\ldots,Y_{2n},\ldots,Y_{m1},\ldots,Y_{mn}) = 0,$$
$$\alpha \in Q_{r,m}, \quad \beta \in Q_{r,n}. \qquad (2)$$

Using Theorem 1, we can conclude that there exists polynomials $G_i \in K[X_\alpha \mid \alpha \in \Gamma_{m,n}]$, $i = 1,\ldots,p$, such that $\mathscr{L}(z) = \sum_{\alpha \in \Gamma_{m,n}} b_\alpha e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)}$ is decomposable if and only if $G_i(b_\alpha \mid \alpha \in \Gamma_{m,n}) = 0$, $i = 1,\ldots,p$. Then $\mathscr{L}$ is a linear operator that preserves decomposability if and only if the polynomials

$$G_i\big(H'_\alpha(Y_{11},\ldots,Y_{1n},Y_{21},\ldots,Y_{2n},\ldots,Y_{m1},\ldots,Y_{mn}) \big| \alpha \in \Gamma_{m,n}\big)$$

are equal to zero for all $m \times n$ matrices $A = (a_{ij})$ of rank less than or equal to r for all $i \in \{1,\ldots,p\}$. But, assuming that $\mathscr{L}$ preserves r-decomposability, we know that $G_i(H'_\alpha(a_{11},\ldots,a_{1n},a_{21},\ldots,a_{2n},\ldots,a_{m1},\ldots,a_{mn})) = 0$, $i = 1,\ldots,p$, when for some $\gamma \in Q_{r,m}$ and $\omega \in Q_{r,n}$ we have $D_{\gamma,\omega}(a_{11},\ldots,a_{1n},a_{21},\ldots,a_{2n},\ldots,a_{m1},\ldots,a_{mn}) \neq 0$. But the last statement means that the polynomial functions $G_i(H'_\alpha(Y_{11},\ldots,Y_{1n},Y_{21},\ldots,Y_{2n},\ldots,Y_{m1},\ldots,Y_{mn}))$, $i = 1,\ldots,p$, are zero over the nonempty open set W (in the affine Zariski topology) which is the complement of the affine closed set with defining polynomials

$$\big(D_{\alpha,\beta}(Y_{11},\ldots,Y_{1n},Y_{21},\ldots,Y_{2n},Y_{m1},\ldots,Y_{mn})\big)_{\alpha \in Q_{r,m},\, \beta \in Q_{r,n}}.$$

Thus [20, p. 25], $A \to G_i(H'_\alpha(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn}))$ is the zero function over W, $i = 1, \dots, p$, and $\mathscr{L}$ preserves t-decomposability, $1 \leqslant t \leqslant r$.

COROLLARY. *Let G be a group, and $\mathscr{L}$ be a linear operator from $L(\otimes^m V, \otimes^m V)$. The linear operator $\mathscr{L}$ is an m-decomposability preserver with respect to $V_1(G)$ if and only if it preserves symmetrized decomposability.*

FINAL REMARKS

In the Grassmann space the notions of decomposability and m-decomposability coincide, since the only nonzero decomposable tensors are the m-decomposable tensors. In this case Theorem 2 is trivial.

If we combine Theorem 1 and the results contained in [20] with the proof of Theorem A, we get a family of homogeneous polynomials with a role equivalent to that of the Plücker polynomials for the decomposable tensors of the symmetry classes of tensors $V_1(G)$.

It is known [6] that the result in the Lemma is generalizable to symmetry classes of tensors $V(c)$ satisfying $\sum_{\sigma \in S_m} c(\sigma) \neq 0$. Thus Theorem 2 is still valid for these symmetry classes of tensors.

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After submitting the paper I was notified that M. H. Lim had independently obtained Theorem 1.

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